Image Sampling with Quasicrystals

Mark GRUNDLAND †, Jiří PATERA‡, Zuzana MASÁKOVÁ* and Neil A. DODGSON †

† Computer Laboratory, University of Cambridge, UK

‡ Centre de Recherches Mathématiques, Université de Montréal, Canada

* Department of Mathematics & Doppler Institute FNSPE, Czech Technical University in Prague, Czech Republic

1 Introduction

An important mathematical tool in a variety of computer graphics applications are non-periodic tilings. They have proven especially useful in the design of sampling algorithms, where they serve to direct the spatial distribution of rendering primitives by enforcing spatial uniformity while precluding regular repetition. Recently, Wang tilings, Penrose tilings, Socolar tilings and polyominoes have been used to generate point sets for non-periodic sampling, usually cunstructed by matching rules or hierarchical substitution. We present the cut-and-project method of generating quasicrystals as an alternative algebraic approach to the production of non-periodic tilings and point sets usable in computer graphics applications. This algebraic approach has the advantages of being:

- straightforward to implement,
- easy to calculate,
- readily amenable to rigorous mathematical analysis,
- directly extended to higher dimensions as well as adaptive sampling applications.

Originally, the theory of quasicrystals was motivated by solid state physics as a model of the non-periodic geometric structures that describe the symmetries of a new kind of long-range atomic order discovered in certain metallic alloys [Shechtman, et al., 1984]. They displayed pentagonal and decagonal rotational symmetries, which cannot occur in any periodic arrangement. Suitable model was provided by cut-and-project sets based on the geometry and algebra of the golden ratio $\tau = \frac{1}{2}(1+\sqrt{5})$, as described for example in [Moody and Patera, 1993].

In this work, we present an evaluation of non-periodic image sampling using cut-and-project quasicrystals, as compared to other non-adaptive sampling techniques, namely periodic sampling, farthest point sampling, jittered sampling, quasirandom sampling and random sampling.

We base our experimental investigations on our experience with the development of a point-based rendering approach to multiresolution image representation for digital photography based on scattered data interpolation techniques, which has been shown to support a secure and compact image encoding suitable for both photorealistic image reconstruction and non-photorealistic image rendering.

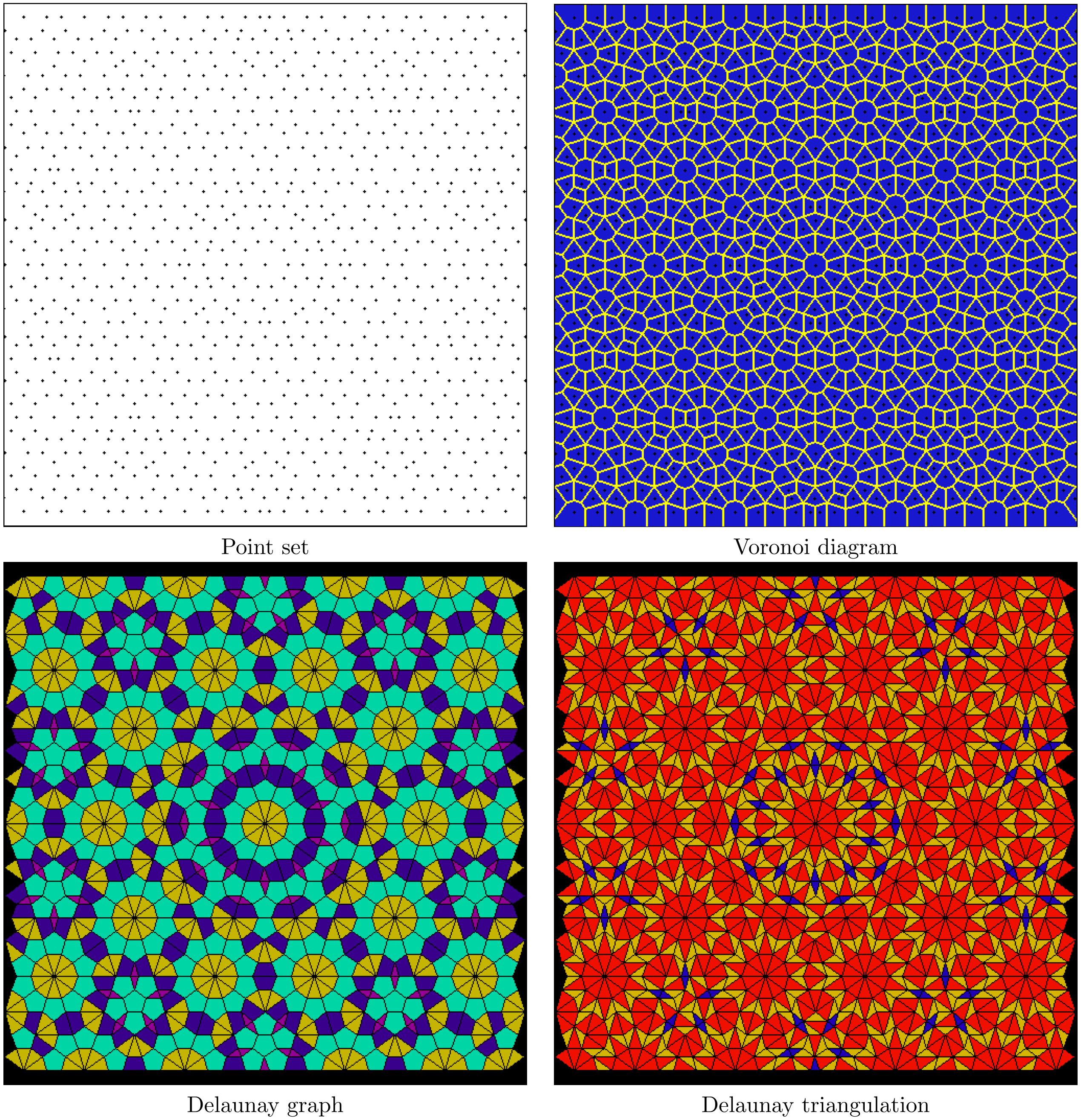


Figure 1: Quasicrystal tilings produced using spatial proximity graphs. In these visualizations, a non-periodic, rotationally symmetric point set (top left) is depicted as a planar tiling induced by a Voronoi diagram (top right), a Delaunay graph (bottom left), and a Delaunay triangulation (bottom right). The depicted set of 1035 points is a fragment of a cut-and-project quasicrystal with decagonal acceptance window.

2 Cut-and-project sets as models for quasicrystals

Some remarkable properties of cut-and-project models of quasicrystals are:

- The Delaunay property, i.e. that quasicrystals are both uniformly discrete and relatively dense, in particular enforcing both a minimal and a maximal distance between each sample site and its closest neighboring site.
- The finite type property, i.e. that there is only a finite number of configurations of given size.
- The repetitivity property, i.e. that each fragment is repeated infinitely many times in the mosaic.

The model we use employs the standard root lattice of the non-crystallographic Coxeter group H_2. To produce a 2D cut-and-project quasicrystal, a 4D periodic lattice is projected on a suitable 2D plane that is irrationally oriented with respect to the lattice. The choice of points to be projected is given by a bounded region, called the acceptance window.

For the definition of 1D cut-and-project quasicrystals we need:

Golden ratio:

$\tau = \frac{1}{2}\left(1+\sqrt{5}\right)$ and its conjugate $\tau' = \frac{1}{2}\left(1-\sqrt{5}\right)$ are the solutions of $x^2 = x+1$.

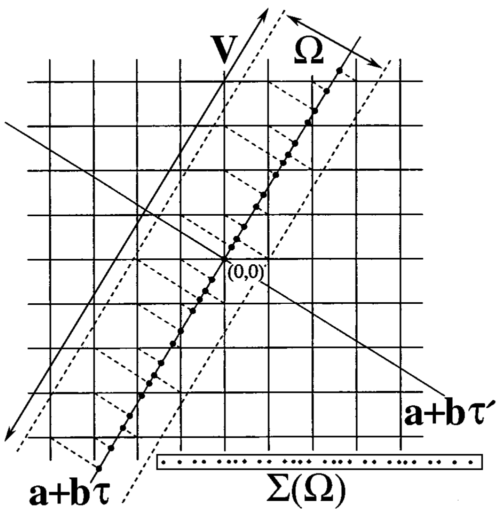
Golden integers:

$\mathbb{Z}[\tau] = \{a + b\tau \mid a, b \in \mathbb{Z}\}$ is an Euclidean domain that is dense in $\mathbb{R}$.

1D cut-and-project quasicrystal

$\Sigma_\Omega = \{a + b\tau \in \mathbb{Z}[\tau] \mid a + b\tau' \in \Omega\}$, where $\Omega = [c, d)$.

The construction is illustrated on the figure.



The notions used for the definition of 2D cut-and-project quasicrystals are:

Star basis

$e_1 = (0,1)$, $e_2 = (-\frac{1}{2}\tau, \frac{1}{2}\sqrt{3-\tau})$, resp. $e_1^* = (0,1)$, $e_2^* = (\frac{1}{2}(\tau-1), -\frac{1}{2}\sqrt{2+\tau})$, are the simple roots of the Coxeter group H_2.

Golden integer lattice:

$M = \mathbb{Z}[\tau]e_1 + \mathbb{Z}[\tau]e_2 = \{(a_1 + b_1\tau)e_1 + (a_2 + b_2\tau)e_2 \mid a_1, b_1, a_2, b_2 \in \mathbb{Z}\}$ is a $\mathbb{Z}[\tau]$-module that is dense in $\mathbb{R}^2$.

2D cut-and-project quasicrystal:

$\Sigma_\Omega = \{xe_1 + ye_2 \in M \mid x'e_1^* + y'e_2^* \in \Omega\}$, where Ω is a bounded acceptance window.

3 Evaluation

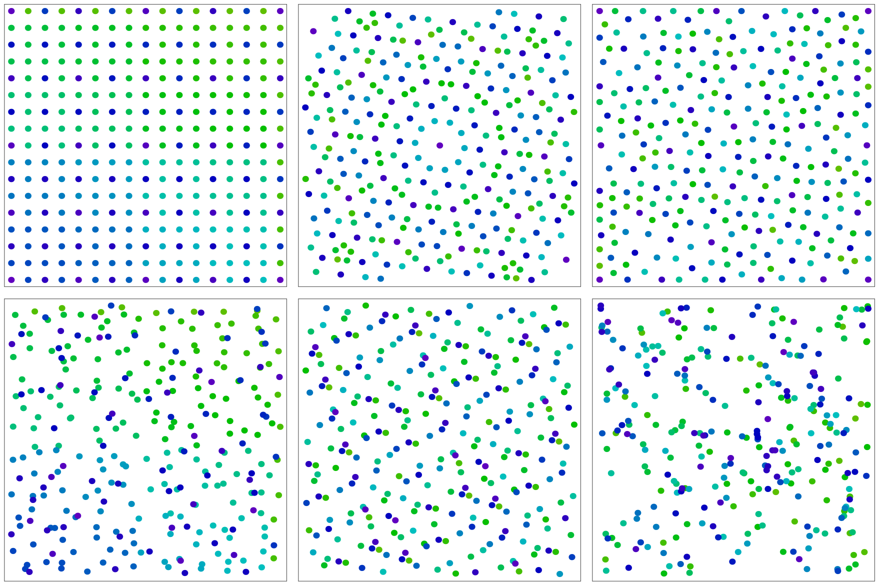
We compared quasicrystal sampling to a number of standard non-adaptive image sampling strategies, namely:

- **Periodic sampling** aims for global regularity. Our implementation relies on a square lattice refined in scan line order.
- **Quasicrystal sampling** aims for local regularity. Our implementation relies on the cut and project method applied using the golden ratio.
- **Farthest point sampling** aims for spatial uniformity. Our implementation relies on the principle of progressively sampling at the point of least information, placing each new sample site at the point farthest from any preceding sample site, which is necessarily a vertex of the Voronoi diagram of the preceding sample sites.
- **Jittered sampling** aims for local variability. Our implementation relies on a random displacement of a square lattice refined in scan line order.
- **Quasirandom sampling** aims for low discrepancy. Our implementation relies on the Halton sequence.
- **Random sampling** aims for global variability. Our implementation relies on a uniform distribution.

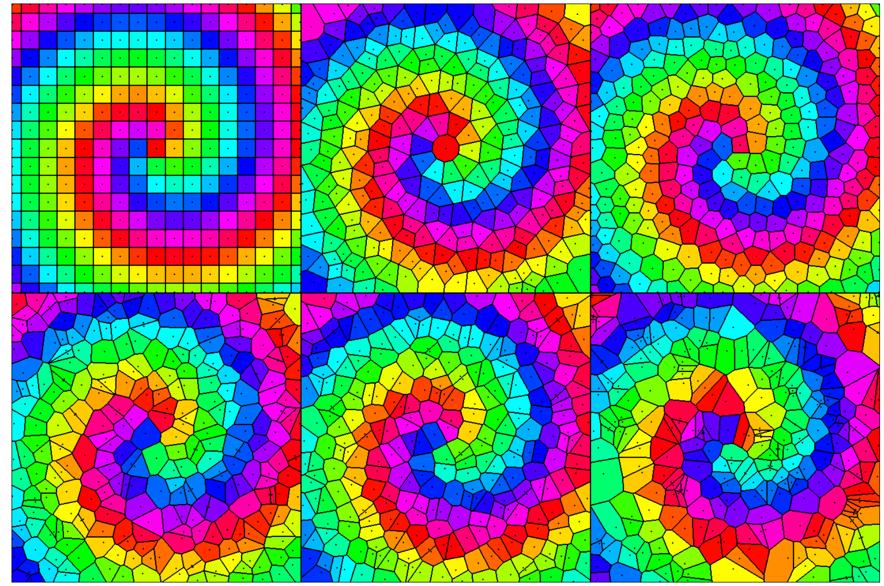
We used seven criteria known to affect the visual quality of photorealistic image reconstruction and non-photorealistic image rendering:

- **Accurate reconstruction** requires the rendition to faithfully represent the likeness of the original image. This objective is a necessary but not sufficient condition of success in both photorealistic and non-photorealistic image rendering.
- **Progressive refinement** requires the sample sites to smoothly fill the available space, avoiding abrupt changes in appearance as new sample sites are sequentially added to the rendition. This objective serves to enable a multiresolution image representation to support progressive rendering of compressed images based on an incremental sampling of the image data.
- **Uniform coverage** requires the sample sites to be evenly distributed regardless of position, avoiding configurations that place sample sites too close or too far from their nearest neighbors.
- **Isotropic distribution** requires the sample sites to be evenly distributed regardless of orientation, avoiding configurations that align sample sites along globally or locally preferred directions.
- **Blue noise spectrum** requires the sample sites to be distributed similarly to a Poisson disk distribution, a random point field conditioned on a minimum distance between the points. A blue noise spectrum is highly desirable in many computer graphics applications, particularly photorealistic image reconstruction.
- **Centroidal regions** require sample sites to be well centered with respect to their Voronoi polygons, approximating a centroidal Voronoi diagram. This objective is popular in non-photorealistic image rendering.
- **Heterogeneous configurations** require sample sites to be placed in a variety of local arrangements, avoiding regularly or randomly repeating the same sampling patterns. While this objective is not traditionally a concern in photorealistic image reconstruction, it helps to prevent non-photorealist image rendering from appearing too perfect, seemingly mechanical and monotonous.

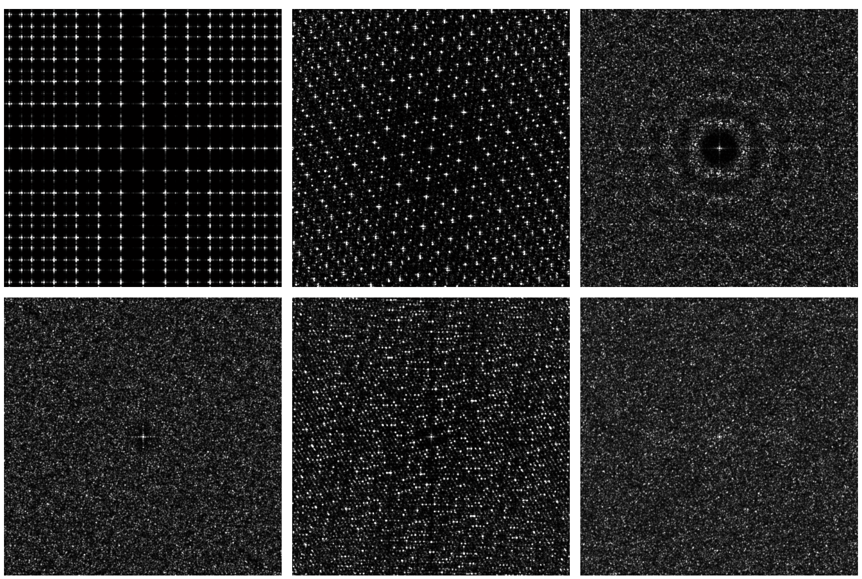
Non-adaptive sampling strategies: periodic (top left), quasicrystal (top center), farthest-point (top right), jittered (bottom left), quasirandom (bottom center), random (bottom right).



Sampling starts with dark blue and finishes with light green sites.



Voronoi diagrams of image sampling strategies.



Fourier power spectra of image sampling strategies.

Sampling Strategies	Accurate Reconstruction	Progressive Refinement	Uniform Coverage	Isotropic Distribution	Blue Noise Spectrum	Centroidal Regions	Heteroeneous Configurations
Periodic	★★★★	★	★★★★	★	★	★★★★	★
Quasicrystal	★★★	★★★★	★★★	★	★	★★★	★★
Farthest Point	★★★★	★★★★	★★★★	★★★	★★★★	★★★	★★
Jittered	★	★	★★	★★★	★★★	★★	★★★
Quasirandom	★★	★★★	★★	★★	★★	★★	★★★
Random	★	★★	★	★★★★	★	★	★★★★

★★★★ Superior ★★★ Good ★★ Fair ★ Poor

Qualitative evaluation of image sampling strategies.

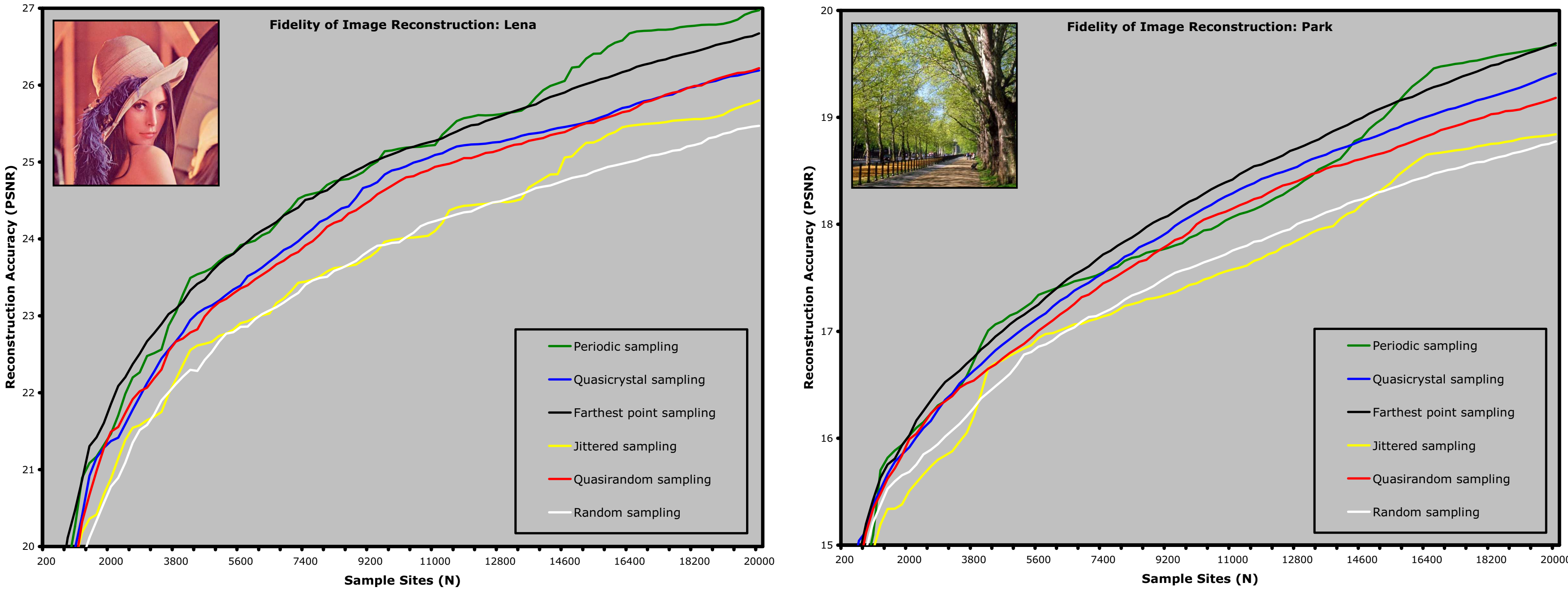


Figure 2: Image reconstruction accuracy graph comparing the non-adaptive sampling strategies.

4 Conclusions

Cut-and-project quasicrystals present new possibilities for image sampling in computer graphics. This non-periodic sampling approach deterministically generates uniformly space-filling point sets, ensuring that sample sites are evenly distributed throughout the image. It offers a useful compromise between predictability and randomness, between the standard periodic sampling and the standard Monte Carlo sampling methods. Although farthest point sampling can generate higher quality sampling patterns, quasicrystal sampling may prove to be more practical in certain contexts because it is much simpler to implement and calculate.

Acknowledgements

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